# The Codimension of Degenerate Pencils

Shmuel Friedland[†]
*Institute of Mathematics*
*The Hebrew University*
*Jerusalem, Israel**
*and*
*Mathematics Research Center*
*University of Wisconsin*
*Madison, Wisconsin*

and

Barry Simon[‡]
*Department of Mathematics*
*California Institute of Technology,*
*Pasadena, California**
*and*
*Department of Mathematics and Physics*
*Princeton University,*
*Princeton, New Jersey*

Submitted by Hans Schneider

ABSTRACT

Let $d_n$ $[d_n(r)]$ denote the codimension of the set of pairs of $n \times n$ Hermitian [really symmetric] matrices $(A, B)$ for which $\det(\lambda I - A - xB) = p(\lambda, x)$ is a reducible polynomial. We prove that $d_n(r) \leqslant n-1$, $d_n \leqslant n-1$ ($n$ odd), $d_n \leqslant n$ ($n$ even). We conjecture that the equality holds in all three inequalities. We prove this conjecture for $n = 2, 3$.

## 1. INTRODUCTION

The calculation of the codimension of various varieties of matrices has been a useful device in understanding various qualitative aspects of eigenvalue perturbation theory. The most famous and the first of these results is the

*Current address.

[†]Research partially supported by Binational Science Foundation under grant 2242/80.

[‡]Research partially supported by NSF under grant MCS-78-01885.




52 Vanderbilt Ave., New York, NY 10017 0024-3795/82/030041+13$02.75

theorem of Wigner and von Neumann [4] which states that the codimension of the variety of $n \times n$ Hermitian matrices with a degenerate eigenvalue in the space of all $n \times n$ Hermitian matrices is independent of $n$ and is equal to three. This implies that "in general," a one-parameter family of Hermitian matrices will not contain a matrix with a degenerate eigenvalue. This result is called in quantum physics "the no-crossing rule".

Consider a pair of complex square matrices $(A, B)$. We identify this pair with the pencil $A(x) = A + xB$, where $x$ belongs to the complex field $C$. A pencil $A + xB$ is called *nondegenerate* if the polynomial

$$p(\lambda, x) = \det(\lambda I - A - xB)$$

is irreducible over $C[\lambda, x]$. If $A(x)$ is a nondegenerate pencil, all eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ of $A(x)$ can be obtained from a single eigenvalue [for example $\lambda_1(x)$] by all possible analytic continuations in $x$. $A(x)$ is a *degenerate* pencil if $p(\lambda, x)$ is a reducible polynomial. "In general" all the eigenvalues of a reducible pencil cannot be obtained from one eigenvalue. (More precisely, all the eigenvalues of a reducible pencil can be generated from a single eigenvalue if and only if $p(\lambda, x) = q(\lambda, x)^m$, where $q(\lambda, x)$ is irreducible and $m \geqslant 2$. It can be shown that such pencils form a proper subvariety in reducible pencils. See for example [2].)

Let $M_n$ $[M_n(r)]$ denote the set of pairs $(A, B)$ of Hermitian [real symmetric] matrices, and let $D_n$ $[D_n(r)]$ be the set of pairs for which $A + xB$ is a degenerate pencil. Since reducibility of $p(\lambda, x) = \sum_{k+j \leqslant n} a_{kj} \lambda^k x^j$, $a_{n0} = 1$, is equivalent to a set of polynomial conditions on $a_{kj}$, clearly $D_n$ and $D_n(r)$ are varieties in $M_n$ and $M_n(r)$. Here we view $M_n$ and $M_n(r)$ as real spaces of dimension $2n^2$ and $n(n+1)$ respectively. In [1] Avron and Simon gave an explicit example of a real symmetric nondegenerate pair $(A, B)$. Thus $D_n$ and $D_n(r)$ are clearly proper subvarieties, so

$$d_n = \operatorname{codim} D_n = \dim M_n - \dim D_n > 0,$$

$$d_n(r) = \operatorname{codim} D_n(r) > 0.$$

In order to understand some results in the analytic theory of bands in state quantum Hamiltonians, Avron and Simon asked for the exact values of $d_n$. By identifying a component of $D_n$ they proved $d_n \leqslant 2n - 2$ and conjectured equality, although they emphasized that the evidence for the equality sign was weak. In this paper we will prove that

$$d_n \leqslant n-1 \qquad (n \text{ odd}), \tag{1.1a}$$

$$d_n \leqslant n \qquad (n \text{ even}), \tag{1.1b}$$

$$d_n(r) \leqslant n-1 \qquad (\text{all } n). \tag{1.1c}$$

Thus, the Avron-Simon conjecture is false if $n \geqslant 3$. We believe that the equality holds for (1.1), in part for reasons explained in [2]. In Section 2 we show

$$d_2 = 2, \qquad d_2(r) = 1,$$

$$d_3 = 2, \qquad d_2(r) = 2.$$

In Section 3 we discuss (1.1) for odd $n$, and in Section 4 for even $n$.

We should mention the relevance of (1.1) to the result of Avron and Simon we are trying to understand. They were interested in a theorem of Kohn [3], who considered a class of pencils $A + xB$, where $A$ and $B$ are specific differential operators, $B$ is fixed, and $A$ depends on a function $V$ periodic on $(-\infty, \infty)$ with period 1. For this particular class, Kohn showed that if $V$ is not constant, then all eigenvalues of $A(x)$ can be obtained from any fixed eigenvalue of $A(x)$ by analytic continuation. In a natural $n$-point difference-equation approximation, $A$ and $B$ are $n \times n$ matrices and $V$ is replaced by an $n \times n$ diagonal matrix. Thus, the intersection of this $n$-dimensional family with $D_n$ is one-dimensional "when $n = \infty$," as can be understood if $d_n \geqslant n-1$ (the constant function plays a special role in Kohn's analysis, so even if $d_n$ were strictly larger than $n-1$, the one dimensional intersection would not be disturbed). If our conjecture is true, one can understand Kohn's result as a specific case of a generic phenomenon.

## 2. THE CASES $n = 2,3$

In the case that $n = 2,3$, $p(\lambda, x) = \det(\lambda I - A - xB)$ is reducible if and only if $p(\lambda, x)$ is divisible by a linear factor $\lambda - a - xb$. Let $\tilde{A} = A - aI$, $\tilde{B} = B - bI$. Then $p(\lambda, x)$ is divisible by $\lambda - a - xb$ if and only if

$$\det(\tilde{A} + x\tilde{B}) = 0. \tag{2.1}$$

LEMMA 2.1. *The pair $(A, B)$ belongs to $D_n$ $[D_n(r)]$ if and only if $A$ and $B$ commute.*

*Proof.* Assume first that $A$ and $B$ commute. Then there exists a unitary matrix $U$ such that $A_1 = U^{-1}AU$ and $B_1 = U^{-1}BU$ are diagonal. So $\det(\lambda I - A - xB) = \det(\lambda I - A_1 - xB_1) = (\lambda - a_1 - xb_1)(\lambda - a_2 - xb_2)$. Vice versa, suppose that $\det(\lambda I - A - xB)$ splits to a product of two linear factors. Let $\tilde{A}$ and $\tilde{B}$ be defined as above. It is enough to show that $\tilde{A}$ and $\tilde{B}$ commute. By changing basis we can suppose that

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} b_1 & c \\ \bar{c} & b_2 \end{pmatrix},$$

since $\det \tilde{A} = 0$. Then (2.1) becomes $b_1\alpha = 0$, $b_1 b_2 - |c|^2 = 0$. If $\alpha = 0$, then $\tilde{A} = 0$, so $[A, B] = AB - BA = 0$ trivially. If $\alpha \neq 0$, then $b_1 = 0$ and the second equality implies $c = 0$. That is, $\tilde{B}$ is diagonal and $\tilde{A}$ and $\tilde{B}$ commute. ∎

THEOREM 2.2. *Let $D_2$ $[D_2(r)]$ be pairs of degenerate $2 \times 2$ Hermitian (real symmetric) matrices. Then*

$$\begin{aligned} \dim D_2 &= 6, & d_2 &= 8 - 6 = 2 \\ \dim D_2(r) &= 5, & d_2(r) &= 6 - 5 = 1. \end{aligned} \tag{2.2}$$

*Proof.* According to Lemma 2.1, $A, B \in D_2$ [or $D_2(r)$] and if and only if $[A, B] = 0$. Either $A = aI$ and $B$ is arbitrary, leading to a component of dimension 5 [or 4], or $A$ is arbitrary and $B = b_1 I + b_2 A$, leading to a component of dimension 6 [or 5]. ∎

REMARKS.

(1) The codimension-$(2n-2)$ component found by Avron and Simon consists of pairs $(A, B)$ with a common invariant subspace. For $n = 2$ all degenerate pencils have a common invariant subspace, which explains why they got the correct answer in that case.

(2) Let $M_n(c)$ denote the complex space of all $(A, B)$ where $A$ and $B$ are $n \times n$ complex symmetric matrices. Denote by $d_n(c)$ the complex codimension of the degenerate pencils. Then

$$d_2(c) = 1.$$

The extra condition on $D_2$ comes from the fact that the single condition $|c|^2=0$ (which is replaced by $c^2=0$ in the complex symmetric case) implies $\operatorname{Re} c=0$ and $\operatorname{Im} c=0$. This example reveals the extra difficulty in computing dimensions of polynomial varieties in $R^n$ as opposed to $C^n$.

**Theorem 2.3.** *Let $D_3$ $[D_3(r)]$ be pairs of degenerate $3\times 3$ Hermitian (real symmetric) matrices. Then*

$$\begin{aligned} \dim D_3 &= 16, \qquad & d_3 &= 18-16=2, \\ \dim D_3(r) &= 10, \qquad & d_3(r) &= 12-10=2. \end{aligned} \tag{2.3}$$

*Proof.* Let $(A,B)\in D_3$ $[D_3(r)]$. As in the case $n=2$, $\det(\lambda I - A - xB)$ has a linear factor, so (2.1) holds. After a change of basis,

$$A=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, \qquad \tilde{B}=\begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix}, \quad \bar{\tilde{b}}_{ij}=\tilde{b}_{ji}.$$

Let us assume the generic case, i.e., $\alpha_1 \neq \alpha_2$, $\alpha_1\alpha_2 \neq 0$. Then (2.1) becomes

$$\det(\tilde{B})=0, \qquad \alpha_1\alpha_2\tilde{b}_{11}=0,$$

$$\alpha_1\left(\tilde{b}_{11}\tilde{b}_{33}-|\tilde{b}_{13}|^2\right)+\alpha_2\left(\tilde{b}_{11}\tilde{b}_{22}-|\tilde{b}_{12}|^2\right)=0.$$

Since $\alpha_1\alpha_2 \neq 0$, the equalities reduce to

$$\det \tilde{B}=0, \qquad \tilde{b}_{11}=0, \qquad \alpha_1|\tilde{b}_{13}|^2+\alpha_2|\tilde{b}_{12}|^2=0. \tag{2.4}$$

The equations (2.4) give rise to two distinct components. For $\alpha_1\alpha_2>0$ the last equality in (2.4) implies $\tilde{b}_{13}=\tilde{b}_{12}=0$. In that case (2.4) reduces to $\tilde{b}_{11}=\tilde{b}_{12}=\tilde{b}_{13}=0$. Taking into account that $\alpha_3=0$ ($A$ has zero eigenvalue), we see that we have lost 6 real parameters (in the real case we lost 4 real parameters). By letting $A=\tilde{A}+aI$, $B=\tilde{B}+bI$ we recover two real parameters. If we denote this component of $D_3$ $[D_3(r)]$ by $A_3$ $[A_3(r)]$, then we get

$$\begin{aligned} \operatorname{codim} A_3 &= 4, \qquad & \dim A_3 &= 18-4=14, \\ \operatorname{codim} A_3 &= 2, \qquad & \dim A_3 &= 12-2=10. \end{aligned} \tag{2.5}$$

However, if $\alpha_1\alpha_2<0$, then the last equation in (2.4) eliminates only one real parameter. In that case the conditions (2.4) reduce 3 real parameters in $B$. If we denote the second component of $D_3$ $[D_3(r)]$ by $B_3$ $[B_3(r)]$, then the above arguments show

$$\begin{aligned} &\operatorname{codim} B_3=2, \qquad \dim B_3=18-2=16,\\ &\operatorname{codim} B_3(r)=2, \qquad \dim B_3(r)=12-2=10, \end{aligned} \tag{2.6}$$

It is left to consider the case where $A$ has a multiple eigenvalue. Then by the Wigner–von Neumann theorem $\operatorname{codim} W_n=3$, and one can easily show that $\operatorname{codim} W_n(r)=2$. Clearly

$$\operatorname{codim}(W_3\cap D_3)>3, \qquad \operatorname{codim}\big[W_3(r)\cap D_3(r)\big]>2. \tag{2.7}$$

This establishes the equalities (2.3). ■

REMARK. The $A_3$ component is precisely the one found by Avron and Simon. It has codimension $4=2n-2$, as they computed.

## 3. ODD $n$

To get lower bounds on $\dim D_n$ we need only to find a component of $D_n$ with the required dimension. While not every component of $D_n$ will have a linear factor in $p(\lambda,x)=\det(\lambda I-A-xB)$ when $n\geqslant 4$, according to [2] the component of $D_n$ with the highest dimension is the component for which $p(\lambda,x)$ has a linear factor. Motivated by (2.1) and the proof of Theorem 2.3, we try $A$ with an index $[n/2]$. By considering the matrices $Q\tilde{A}Q^t$, $Q\tilde{B}Q^t$, we may assume that

$$A_0=\operatorname{diag}(0,1,-1,1,-1,\dots,1,-1), \tag{3.1}$$

where $n=2m+1$.

PROPOSITION 3.1. *Let $A=A_0$ as in* (3.1). *Then the dimension of the set $B$ of Hermitian matrices $B$ with* $\det(A_0+xB)=0$ *is of dimension $n^2-n$ at least.*

Accepting this result for the moment, let us prove

THEOREM 3.2. *Let $D_n$ be the set of $n \times n$ Hermitian degenerate pairs. Then*

$$\dim D_n \geq 2n^2 - (n-1), \qquad d_n \leq n-1$$

*if $n$ is odd.*

*Proof.* Let $A$ be a generic matrix with $2m+1$ distinct eigenvalues. Let $\lambda_1(A) > \lambda_2(A) > \cdots > \lambda_{2m+1}(A)$ be the eigenvalues of $A$. Then there exists a unitary matrix $U(A)$ which can be chosen to depend smoothly on $A$ in some neighborhood of a $A_1$, with distinct eigenvalues, such that

$$U(A)^*AU(A) = \operatorname{diag}(\lambda_{m+1}(A), \lambda_1(A), \lambda_{2m+1}(A), \ldots, \lambda_m(A), \lambda_{m+2}(A)).$$

Define

$$D(A) = \operatorname{diag}(d_1(A), \ldots, d_{2m+1}(A)),$$

$$d_1(A) = 1,$$

$$d_{2i}(A) = [\lambda_i(A) - \lambda_{m+1}(A)]^{1/2},$$

$$d_{2i+1}(A) = [\lambda_{m+1}(A) - \lambda_{2m-i+2}(A)]^{1/2}, \qquad i = 1, 2, \ldots, m.$$

Let $B$ be any matrix satisfying $\det(A_0 + xB) = 0$, where $A_0$ is given by (3.1). Put

$$C = U(A)D(A)BD(A)U(A)^* + cI. \tag{3.2}$$

Then

$$\det\{A + xC - [\lambda_{m+1}(A) - cx]I\} = \det[U(A)D(A)(A_0 + xB)D(A)U^*(A)]$$
$$= 0$$

That is, $\det(\lambda I - A - xC)$ has a linear factor $\lambda - \lambda_{m+1}(A) - cx$. A direct count of the parameters shows that this component of degenerate pencils has at least the dimension $2n^2 - (n-1) = n^2 + n^2 - n + 1$. ∎

Let

$$\det(A_0 + xB) = \sum_{j=1}^{n} q_j(B)x^j. \tag{3.3}$$

Thus, the condition $\det(A + xB) \equiv 0$ is equivalent to $n$ polynomial equations

$$q_j(B) = 0, \qquad j = 1, \dots, n. \tag{3.4}$$

Therefore over the complex numbers this algebraic variety has codimension $n$ at most. However, since $B$ is taken to be Hermitian, we have to show explicitly that the codimension of (3.4) is at most $n$. It is easy to see that $q_1(B) = b_{11}$. So (3.4) yields that $b_{11} = 0$. The matrix $B$ is parametrized by $n^2 - 1$ real numbers $\xi_{ij} = \mathrm{Re}\, b_{ij}$, $\eta_{ij} = \mathrm{Im}\, b_{ij}$ for $i < j$ and $\xi_{ii} = b_{ii}$ for $1 < i$. For simplicity of notation we denote these parameters by $y_1, \dots, y_{n^2-1}$, and we view the numbers $q_2(B)$, $q_n(B)$ as the elements of $R^{n-1}$. Thus the equality (3.3) ($b_{11} = 0$) defines a polynomial map $F\colon R^{n^2-1} \to R^{n-1}$, $F = (F_1, \dots, F_{n-1})$. If we can find $y^{(0)}$ with $F(y^{(0)}) = 0$ such that

$$\operatorname{rank} \frac{\partial F_\alpha}{\partial y_\beta}(y^0) = n - 1,$$

then by the implicit-function theorem $\{y \mid F(y) = 0\} \cap$(a neighborhood $y_0$) is a smooth manifold of dimension $n^2 - n$. Obviously, it suffices to find $n - 1$ independent parameters $z_1, \dots, z_{n-1}$ such that the square matrix $\dfrac{\partial F_\alpha}{\partial z_\beta}(y^0)$ is nonsingular, i.e., the kernel of this matrix is trivial.

Now let $P_i(x)$ be the polynomial $P_i(x) = (\partial/\partial z_i)[\det(A_0 + xB)]$. Then the corresponding kernel is trivial if and only if $P_1(x), \dots, P_{n-1}(x)$ are linearly independent. Thus we seek $n$ Hermitian matrices $B_0, B_1, \dots, B_{n-1}$ (the last $n - 1$ matrices linearly independent) such that $\det(A_0 + xB_0) \equiv 0$, and

$$P_i(x, z) = \frac{\partial}{\partial z_i} \det(A_0 + xB_0 + xz_iB_i), \qquad i = 1, \dots, n - 1, \tag{3.5}$$

are linearly independent for $z = 0$. To this end we need the following observation.

LEMMA 3.3. *Let $C=(c_{ij})$ be an $n\times n$ matrix with $c_{ij}=0$ if $i\geq 2$, $j\geq 2$, and $i\neq j$. Suppose that $c_{jj}\neq 0$ for $j\geq 2$. Then*

$$\det C = -\prod_{j=2}^{n} c_{jj}\left(\sum_{j=\alpha}^{n} c_{jj}^{-1}c_{1j}c_{j1}-c_{11}\right). \tag{3.6}$$

*Proof.* For $j=2,\dots,n$, from the first row, subtract the $j$th row multiplied by $c_{1j}c_{jj}^{-1}$. The result is a lower triangular matrix with diagonal elements $c_{11}-\sum_{j=2}^{n}c_{jj}^{-1}c_{1j}c_{j1}, c_{22},\dots,c_{nn}$. ■

*Proof of Proposition 3.1.* We will let $B_0$ be the Hermitian matrix of the form given by Lemma 3.3 having the diagonal elements $0,1,-1,2,-2,\dots,m,-m$ and the first row $0,1,\dots,1$. By the above lemma

$$\det(A_0+xB_0)=-x^2\prod_{j=1}^{m}(1+jx)(-1-jx)\left[\sum_{j=1}^{m}\frac{1}{1+jx}+\frac{1}{-1-jx}\right]\equiv 0.$$

Let $\sum_{i=1}^{2m}z_iB_i$ be a real symmetric matrix satisfying the conditions of Lemma 3.3 with the diagonal elements $0,z_1,0,z_2,\dots,z_m,0$ and the first row $0,z_{m+1},0,z_{m+2},\dots,z_{2m},0$.

By (3.6)

$$\det\left(A+xB_0+\sum_{i=1}^{n-1}xz_iB_i\right)=Q(x,z)\left[\sum_{j=1}^{m}\frac{(1+z_{m+j})^2}{1+x(j+z_j)}-\frac{1}{1+xj}\right],$$

where

$$Q(x,z)=-x^2\prod_{j=1}^{m}\left(1+x(j+z_j)\right)(-1-xj).$$

So

$$P_j(x,0)=-\frac{Q(x,0)x}{(1-xj)^2},\qquad P_{j+m}=\frac{2Q(x,0)}{1+xj}.$$

The equality

$$\sum_{i=1}^{2m} \alpha_i P_i(x,0) \equiv 0$$

implies

$$\sum_{j=1}^{m} -\alpha_j x(1+xj)^{-2} + 2\alpha_{j+m}(1+xj)^{-1} \equiv 0.$$

Multiplying this identity by $(1+xj)^2$ and letting $x=-1/j$, we deduce that $\alpha_j=0$ for $j=1,\dots,m$. A similar argument implies that $\alpha_{j+m}=0$, $j=1,\dots,m$. This establishes the linear independence of $P_1(x,0),\dots,P_{2m}(x,0)$ and completes the proof of the theorem. ∎

Notice that in the above proof all the matrices involved were real symmetric. That is, the set of all real symmetric matrices $B$ satisfying $\det(A_0+xB)\equiv 0$ is at most of codimension $n$. Then for any generic real symmetric matrix $A$ we construct the matrix $C$ given (3.2), where $U(A)$ is a real orthogonal matrix. As before, we conclude that the codimension of all pencils $A+xC$ such that $\det(\lambda I - A - xC)$ has a linear factor has at most codimension $n-1$.

THEOREM 3.4. *Let $D_n(r)$ be the set of $n\times n$ real symmetric degenerate pairs. Then $d_n(r)\leqslant n-1$, $\dim D_n(r)\geqslant n^2+1$ if $n$ is odd.*

## 4. EVEN $n$

The results of Section 2 show that for an even $n$ there is a distinction between the codimension of real symmetric and Hermitian degenerate pencils. A technical reason for that is that if a singular Hermitian matrix $A$ has the equal number of positive and negative eigenvalues, then $A$ has at least a double zero eigenvalue. According to Wigner and von Neumann, the codimension of all such Hermitian matrices is 4. However if we consider all real symmetric matrices with a double zero eigenvalue, the codimension of this set is 3.

In order to prove the inequalities (1.1b) and (1.1c) for an even $n$ we must give the correct analog to the key result of Section 3—Proposition 3.1. The explanation we gave above suggests the "right" form of $A_0$ for $n=2m+2$:

$$A_0=\text{diag}(0,0,1,-1,1,-1,\dots,1,-1). \tag{4.1}$$

PROPOSITION 4.1. *Let $A_0$ be as defined above. Then the dimension of the set $B$ $[B(r)]$ of Hermitian [real symmetric] matrices satisfying* $\det(A_0+xB)\equiv 0$ *is of codimension $n-1$ at most.*

*Proof.* Consider the equality (3.3). Clearly

$$q_1(B)=0, \qquad q_2(B)=(-1)^m(b_{11}b_{22}-|b_{12}|^2). \tag{4.2}$$

Thus if we restrict ourselves to all $B=(b_{ij})$ such that

$$b_{11}b_{22}=|b_{12}|^2, \tag{4.3}$$

then

$$\det(A_0+xB)=\sum_{j=3}^{n} q_j(B)x^j. \tag{4.4}$$

Choose $B_0$ to be a Hermitian matrix satisfying the assumptions of Lemma 3.3, with the diagonal elements $b_{11}^{(0)}, b_{22}^{(0)}, 1, -1, \dots, m, -m$ and the first row $b_{11}^{(0)}, b_{12}^{(0)}, 1, 1, \dots, 1$. Here we assume that $b_{11}^{(0)}b_{22}^{(0)}=|b_{12}^{(0)}|^2>0$.

Again using Lemma 3.3, we easily deduce $\det(A_0+xB_0)=0$.

Now let $\sum_{i=1}^{2m} z_iB_i$ be a real symmetric matrix satisfying the conditions of Lemma 3.3 with the diagonal elements $0,0,z_1,0,z_2,\dots,z_m,0$ and the first row $0,0,z_{m+1},\dots,z_{2m},0$. The calculations carried out in the previous section show that the polynomials $P_1(x,0),\dots,P_{2m}(x,0)$ are linearly independent. That is, the set of all Hermitian matrices $B=(b_{ij})$ satisfying $b_{ij}=b_{ij}^{(0)}$ for $1\le i,j\le 2$ and the equality $\det(A_0+xB)\equiv 0$ is of codimension $4+2m$ at most. However, since we allowed to choose $b_{ij}^{(0)}$, $1\le i,j\le 2$, free within the restriction (4.3), the codimension of $B$ is at most $2m+1$. In the real symmetric case we choose $b_{12}^{(0)}$ to be real, and we deduce as before that the codimension of $B(r)$ is at most $n-1$. ∎

THEOREM 4.2. *Let $D_n$ $[D_n(r)]$ be the set of $n \times n$ Hermitian [real symmetric] degenerate pairs. Then*

$$d_n \leqslant n, \qquad \dim D_n \geqslant 2n^2 - n,$$

$$d_n(r) \leqslant n-1, \qquad \dim D_n(r) \geqslant n^2+1$$

*if n is even.*

*Proof.* Let $A$ be a Hermitian matrix with a double middle eigenvalue

$$\lambda_1(A) > \cdots > \lambda_m(A) > a = \lambda_{m+1}(A) = \lambda_{m+2}(A) > \cdots > \lambda_{2m+2}(A) \tag{4.5}$$

The Wigner–von Neumann result implies that the codimension of such sets of matrices is 3. Let

$$U(A)^*AU(A) = \operatorname{diag}(a, a, \lambda_1(A), \lambda_{2m+2}(A), \dots, \lambda_m(A), \lambda_{m+3}(A)),$$

$$D(A) = \operatorname{diag}(d_1(A), \dots, d_{2m+2}(A)),$$

$$d_1(A) = d_2(A) = 1,$$

$$d_{2i+1}(A) = [\lambda_i(A) - a]^{1/2},$$

$$d_{2i+2}(A) = [a - \lambda_{2m+3-i}(A)]^{1/2}, \qquad i = 1, \dots, m.$$

Let $B$ be any matrix satisfying $\det(A_0 + xB) \equiv 0$. Define $C$ by (3.3). As in the proof of Theorem 3.2, $\det(\lambda I - A - xC)$ has a linear factor. So the codimension of all pairs $(A, C)$ is at most $3+(n-1)-1 = n+1$. Finally we consider all pencils of the form $(A + \alpha C, C)$, where $\alpha$ is a real parameter and $(A, C)$ is the pencil described above. Clearly $(A + \alpha C, C)$ is also a degenerate pencil. It is left to show that the set of all degenerate pencils of the form $(A + \alpha C, C)$ is not contained in the original set $(A, C)$. To this end it is enough to show that $A + \alpha C$ has $n$ distinct eigenvalues for some $A$ and $\alpha$. By the definition of $C$, $A + \alpha C - (a+c)I$ is equivalent to the matrix $A_0 + \alpha B$, where $\det(A_0 + \alpha B) = 0$. Choose $B = B_0$ as in the proof of Proposition 4.1.

Since $b_{11}^{(0)} b_{22}^{(0)} = |b_{12}^{(0)}|^2 > 0$ for a small $\alpha \neq 0$, $A_0 + \alpha B$ will have only one eigenvalue which is equal to zero. Therefore $A + \alpha C$ has pairwise distinct eigenvalues. Thus the algebraic set of all degenerate pairs of the form $(A+\alpha C, C)$ has at least one codimension less than the set $(A, C)$. That is, the codimension of $(A+\alpha C, C)$ is at most $n$. In the real case the codimension of all real symmetric degenerate pairs of the form $(A+\alpha C, C)$ is $n-1$, since the codimension of all real symmetric matrices with a multiple eigenvalue is 2.

The proof of the theorem is completed. ∎

*We would like to thank the Mathematics Research Center of the University of Wisconsin for its hospitality while this work was done.*

## REFERENCES

1 J. Avron and B. Simon, Analytic properties of band functions, *Ann. Physics* 110:85–101 (1978).
2 S. Friedland, Simultaneous similarity of matrices, to appear.
3 W. Kohn, Analytic properties of Bloch waves and Wannier function, *Phys. Rev.* 115:809–821 (1959).
4 E. Wigner and J. von Neumann, *Phys. Z.* 30:467 (1927); English transl. in *Symmetry in the Solid State* (R. S. Knox and A. Gold, Eds.) Benjamin, New York, 1964, pp. 167–172.

*Received 10 December 1980*